this

$$\omega_0 b_0 = \alpha_* \, gL^2 (\nu\chi)^{-1/2} \tag{4.7}$$

In order to obtain a significant effect of stabilization at values of vibrational velocity which are reasonable from the experimental point of view, one should work with fluids which have the highest possible value of parameter $\sqrt{\nu\chi}$ for sufficiently small characteristic dimensions L. Thus, for a plane layer of water ($\sqrt{\nu\chi} = 0.0038$ cm^2/ s) with a thickness of 2 mm a vibrational velocity $\omega_0 b_0 = 360$ cm/s is obtained from (4.7) as necessary for complete stabilization. This means that at an amplitude of displacement of 2 mm stabilization occurs near 250 Hz. This effect is much more strongly pronounced in fluids with a high value of the parameter $\sqrt{\nu\chi}$ (glycerin, olive oil, some silicone liquids).

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HYPERSONIC FLOW PAST A DELTA WING

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The mode of flow over the windward side of a supersonic leading edge wing is examined. In spite of a number of investigations [1-4], this problem has not been solved correctly. The difficulty consists in the fact that in the flow field behind a strong wave there are regions of homogeneous potential and vortex flows which must be matched with sufficient smoothness.

An analytical theory is developed below for hypersonic flow past a wing with an attached wave. This theory allows to carry out the necessary conjuction of flows.

1. Let us examine the hypersonic flow past a slender conical wing. In the system of coordinates (r, θ, φ) shown in Fig. 1 the equations of conical flows in variables (ψ, φ) [1] take the form

$$\frac{w}{\cos\theta}\frac{\partial u}{\partial\varphi} - v^2 - w^2 = 0, \qquad \frac{w}{\cos\theta}\frac{\partial v}{\partial\varphi} + uv + w^2\,\mathrm{tg}\,\theta = -\frac{1}{\rho\theta_\psi}\frac{\partial p}{\partial\psi}$$

$$\frac{\varkappa}{\varkappa - 1}\frac{p}{\rho} + \frac{u^2 + v^2 + w^2}{2} = C, \qquad \frac{\partial}{\partial\varphi}\frac{p}{\rho^\varkappa} = 0 \qquad (1.1)$$

$$\frac{\partial}{\partial\varphi}\ln(\rho w\theta_\psi) + 2\frac{u}{w}\cos\theta = 0, \qquad w\theta_\varphi = v\cos\theta$$

All variables in the equations are dimensionless and normalized with respect to the free stream velocity U, the density ρ° and the dynamic pressure $\rho^\circ U^2$. The variable ψ satisfies the relationship $v\psi_\theta + w\psi_\varphi \sec\theta = 0$ and represents the stream surface in the conical flow. We shall seek the solution with the assumption that the region of perturbed flow is a strongly compressed thin layer. Then, according to the usual evaluations of the shock layer theory [1], it is convenient to introduce the following transformations:

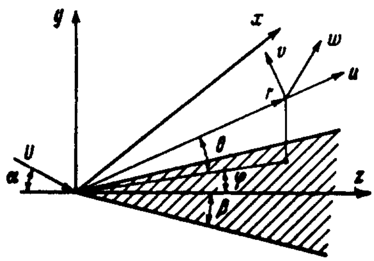

Fig. 1

$$\theta = \varepsilon\bar\theta, \qquad v = \varepsilon\bar v, \qquad \rho = \varepsilon^{-1}\bar\rho \qquad (1.2)$$

With the new variables (the bar on top is omitted) we have the system

$$\frac{w}{\cos\varepsilon\theta}\frac{\partial u}{\partial\varphi} - \varepsilon^2 v^2 - w^2 = 0, \qquad \frac{\varkappa p}{(\varkappa+1)\rho} + \frac{u^2 + \varepsilon^2 v^2 + w^2}{2} = C, \qquad \frac{p^{1/\varkappa}}{\rho} = \delta(\psi)$$

$$\left(\frac{w}{\cos\varepsilon\theta}\frac{\partial u}{\partial\varphi} + uv + w^2\varepsilon^{-1}\,\mathrm{tg}\,\varepsilon\theta\right)\varepsilon = -\frac{1}{\rho\theta_\psi}\frac{\partial p}{\partial\psi} \qquad (1.3)$$

$$\varkappa = \frac{1+\varepsilon}{1-\varepsilon}, \qquad \frac{\partial}{\partial\varphi}[\ln(\rho w\theta_\varphi)] + 2\frac{u}{w}\cos\varepsilon\theta = 0, \qquad w\theta_\varphi = v\cos\varepsilon\theta$$

The variables w and φ are of the same order, therefore the form of this system does not change as the result of normalization of these quantities, and we can take

$$\varphi = \varepsilon^n\bar\varphi \quad (n \geqslant 0)$$

The second equation (1.3) makes it possible to represent the pressure as the sum of two terms

$$p = p_1(\varphi, \varepsilon) + \varepsilon p_2(\psi, \varphi, \varepsilon) \qquad (1.4)$$

Now let us perform a simplification, discarding in (1.3) and in the boundary conditions terms of the order ε^2 and higher. Then the required functions will satisfy the following equations:

$$\frac{\partial u}{\partial\varphi} - w = 0, \quad w\frac{\partial v}{\partial\varphi} + uv + w^2\theta = -\frac{1}{\rho\theta_\psi}\frac{\partial p_2}{\partial\psi}, \quad u^2 + w^2 = \Delta^2(\psi)$$

$$\frac{\partial}{\partial\varphi}\ln(\rho w\theta_\psi) + 2\frac{u}{w} = 0, \qquad w\theta_\varphi = v \qquad (1.5)$$

In this approximation the boundary conditions on the shock wave will assume the form

$$u^* = \cos\alpha\,(\cos\varphi - \varepsilon\theta^*\,\mathrm{tg}\,\alpha), \qquad m_0 = 2/(\varkappa - 1)\,M_\infty^2\sin^2\alpha \qquad (1.6)$$

$$v^* = -[\theta_\varphi^* \cos\alpha \sin\varphi + (1 + m_0)\sin\alpha + \varepsilon\cos\alpha\,(\theta^*\cos\varphi - \\ - \theta_\varphi^*\sin\varphi + \varepsilon\sin\alpha\theta_\varphi^{*2}] \qquad \text{(cont.)}$$

$$w^* = -\cos\alpha\,(\sin\varphi + \varepsilon\theta_\varphi^*\operatorname{tg}\alpha), \qquad p^* = \sin^2\alpha + \varepsilon p_1^*$$

$$\rho^* = (1 + m_0)^{-1}$$

$$p_1^* = \sin 2\alpha\,(\theta^*\cos\varphi - \theta_\varphi^*\sin\varphi) - \sin^2\alpha - M_\infty^{-2}$$

From the flow condition over the surface of the wing, given by the equation $\theta = \varepsilon\theta^\circ(\varphi)$, we have

$$v - w\,\theta_\varphi^\circ(\varphi) = 0 \qquad (1.7)$$

In the derivation of the third equation (1.5), the form of the expression for pressure (1.6) and the fourth equation (1.3) were used. This equation can be written in the form of the following relationship:

$$\frac{1}{\rho} = \frac{1}{\rho'}\left[1 + \frac{\varepsilon}{\sin^2\alpha}\,(p_1^{*\prime} - p_1^* - p_2)\right] \qquad (1.8)$$

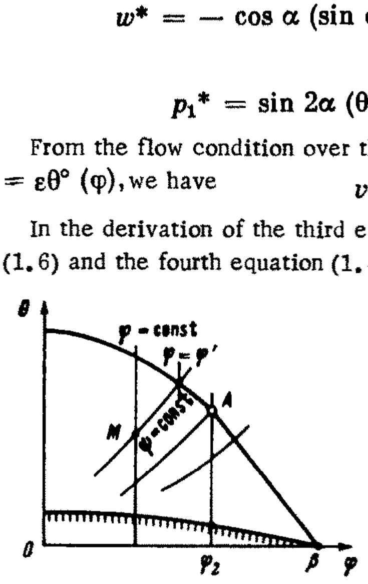

Fig. 2

The primes indicate that the corresponding values are values at the point of intersection of the shock wave with the streamline ($\varphi = \varphi'$ in Fig. 2). The pressure $p_1(\varphi)$ can be determined directly from the value behind the shock wave (1.6). Setting $p_1 = p^*$, then the value p_2 on the shock wave is equal to zero. It should be emphasized that the dependence of p_2 on the parameter ε is determined from the solution of the problem and may not have any analytic representation. However, if the principal term of the expansion of this function is of the order of ε^α, where $\alpha > -1$, the obtained results remain valid, and the discarded terms will be of an order higher than the first.

The system (1.5) permits to determine directly the velocity components u and w. In fact, eliminating u from the first equation and integrating from an arbitrary point to the shock wave along the line $\psi = \mathrm{const}$ (Fig. 2), we find that

$$u = \Delta(\varphi')\cos[\varphi + \alpha(\varphi')], \qquad w = -\Delta(\varphi')\sin[\varphi + \alpha(\varphi')] \qquad (1.9)$$

$$\Delta(\varphi') = \cos^2\alpha - \varepsilon\sin 2\alpha\,(\theta^{*\prime}\cos\varphi' - \theta_\varphi^{*\prime}\sin\varphi') \qquad (1.10)$$

$$\alpha(\varphi') = \varepsilon\operatorname{tg}\alpha\,(\theta_\varphi^{*\prime}\cos\varphi' + \theta^{*\prime}\sin\varphi')$$

Expressions for velocities (1.10) make it possible to represent the integral of the fifth equation of (1.5) in the form

$$\frac{\rho\theta_\psi}{w} = \frac{\rho'\theta_\psi'}{w'} \qquad (1.11)$$

We rewrite the relationship (1.11) in the integral form, replacing the integration with respect to variable ψ by integration with respect to φ'.

As a result we have

$$\theta = \theta^\circ(\varphi) + \int_{\varphi^\circ(\varphi)}^{\varphi'} \frac{\rho' w}{\rho w'}\,\theta_\psi'\psi_{\varphi'}\,d\varphi' \qquad (1.12)$$

Here $\varphi^\circ(\varphi)$ is an arbitrary function corresponding to a streamline on the surface of the wing. It is assumed that the wing is slender and the function $\theta^\circ \sim \varepsilon_1$. The factor $\theta_\psi'\psi_{\varphi'}$ can, according to the last equation of system (1.5) and the second condition

(1.6), be replaced by the expression

$$\theta_{\psi'}\psi_{\varphi'} \equiv \frac{d\theta'}{\partial\varphi'} - \left(\frac{\partial\theta}{\partial\varphi}\right)' = \frac{1}{w'}\left\{(1+m_0)\sin\alpha\left[1+\frac{\varepsilon}{1+m_0}\left(\theta^{*\prime}\cos\varphi' - \theta_\varphi^{*\prime}\sin\varphi'\right)\right]\right\} \qquad (1.13)$$

Substituting (1.8) and (1.13) into (1.12), we find that

$$\theta = \theta^\circ(\varphi) + (1+m_0)\sin\alpha\int\limits_{\varphi^\circ}^{\varphi'}\frac{w}{w'^2}R\,d\varphi'$$

$$R = 1 + \varepsilon\left[\frac{(\theta^{*\prime}\cos\varphi' - \theta_\varphi^{*\prime}\sin\varphi')\operatorname{ctg}\alpha}{1+m_0} + \frac{1}{\sin^2\alpha}(p_1^{*\prime} - p_1^{*} - p_2)\right] \qquad (1.14)$$

From here we obtain the relationship for the shock wave when $\varphi' = \varphi$

$$\theta^* = \theta^\circ(\varphi) + (1+m_0)\sin\alpha\int\limits_{\varphi^\circ}^{\varphi}\frac{w}{w'^2}R\,d\varphi' \qquad (1.15)$$

In addition to the surface of the shock wave, Eq. (1.15) contains also two unknown functions φ° and p_2. To find the first of these, we take advantage of the flow condition (1.7), which, by taking into account (1.5) and (1.14), is reduced to the following form:

$$\frac{d\varphi^\circ}{d\varphi}(w)_{\varphi'=\varphi^\circ} = 0 \qquad (1.16)$$

This equation allows two solutions

$$\varphi^\circ = \mathrm{const}$$

$$z'(\varphi^\circ)\cos\varphi^\circ + z(\varphi^\circ)\sin\varphi^\circ = -\varphi\operatorname{ctg}^2\alpha, \qquad z(\varphi) = \varepsilon\operatorname{ctg}\alpha\theta^*(\varphi) \qquad (1.17)$$

In the exact solution the surface of the wing is the stream surface, therefore $\varphi^\circ = \beta$ (β is the half-angle at the apex of the wing). However, as was pointed out in [1], hypersonic solutions for finite bodies may not satisfy this condition. Without dwelling on the analysis of possible cases, we point out that the theory which is being developed for wings having a region of homogeneous flow near the edges, it is necessary to assume that $\varphi^\circ = \mathrm{const}$. Singularities which can arise in this connection are examined below.

We turn now to finding the second unknown function p_2. First of all we note that the pressure p_2 enters everywhere with the coefficient ε and it is sufficient to determine the principal term. We shall find the pressure p_2 from the second equation (1.5) by integrating this equation over the line $\varphi = \mathrm{const}$ from the shock wave to an arbitrary point. As a result we find that

$$p_2 = \sin\alpha\int\limits_{\varphi^\circ}^{\varphi}\frac{w^3}{w'^2}(\theta_{\varphi\varphi} + \theta)\,d\varphi' \qquad (1.18)$$

If now (1.14) is differentiated twice and the result substituted into (1.18), we shall have

$$p_2 = \sin\alpha(\theta^\circ + \theta^\circ_{\varphi\varphi})\int\limits_{\varphi'}^{\varphi}\frac{w^3}{w'^2}d\varphi' + 2\sin^2\alpha(1+m_0)\Big\{[\cos\varphi(z+z'') + \sin\varphi(z'+z''')]\int\limits_{\varphi'}^{\varphi}\frac{w^3}{w'^2}d\varphi'\int\limits_{\beta}^{\varphi'}\left(\frac{w}{w'^2}\right)_{\varphi'=\xi}d\xi - 2\sin\varphi(z + z'')\int\limits_{\varphi'}^{\varphi}\frac{w^3}{w'^2}d\varphi'\int\limits_{\beta}^{\varphi'}\left(\frac{u}{w'^2}\right)_{\varphi'=\xi}d\xi\Big\} \qquad (1.19)$$

The first term in (1.19) characterizes the influence of curvature of the transverse contour, the second term the influence of curvature of the shock wave. In Eq. (1.19) terms containing the quantities $\varepsilon p_{2\varphi}$ and $\varepsilon p_{2\varphi\varphi}$ are omitted. The basis for this are estimates of discarded terms which with accuracy to an insignificant constant have the following form:

$$\delta_1 = \varepsilon \int_{\varphi'}^{\varphi} \frac{w^3}{w'^2} d\varphi' \int_{\beta}^{\varphi'} \left(\frac{u}{w'^2}\right)_{\varphi'=\xi} p_{2\varphi}\, d\xi, \qquad \delta_2 = \varepsilon \int_{\varphi'}^{\varphi} \frac{w^3}{w'^2} d\varphi' \int_{\beta}^{\varphi} \left(\frac{u}{w'^2}\right)_{\varphi'=\xi} p_{2\varphi\varphi}\, d\xi$$

Assuming that $p_2 \sim a(\varepsilon)$, we have $p_{2\varphi} \sim \varepsilon^{-1/2} a(\varepsilon)$ and $p_{2\varphi\varphi} \sim \varepsilon^{-1} a(\varepsilon)$. Taking into account that $u \sim 1$; $w \sim \varepsilon^{1/2}$ (for $\varphi \lesssim \varphi_2$) and that w' changes its order from 1 to $\varepsilon^{1/2}$, we obtain $\delta_1 \sim \delta_2 \sim \varepsilon a(\varepsilon)$. Consequently, in the region of nonhomogeneous flow where $\varphi \sim \sqrt{\varepsilon}$, the indicated pressure gradients with the corresponding weight functions have a higher order of smallness than the remaining terms of the equations, and the principal term for the pressure p_2 is determined by Eq. (1.19). In the region of homogeneous flow p_2 becomes zero.

Substituting (1.19) into (1.15) we obtain for the surface of the shock wave the following integro-differential equation

$$z + \delta\,[2L_2\,(z\cos\varphi - z'\sin\varphi) - L_1] + 2\varepsilon\,(z^\circ + z^{\circ\prime\prime})\,L_3 - z^\circ =$$

$$= 2\varepsilon\delta\,\{(z + z'')\,[2\sin\varphi L_4 - \cos\varphi L_5] - (z' + z''')\,\sin\varphi\, L_5\}$$

$$z = \varepsilon \operatorname{ctg}\alpha\,\theta^*(\varphi), \qquad z^\circ = \varepsilon \operatorname{ctg}\alpha\,\theta^\circ(\varphi), \qquad \delta = \varepsilon\cos\alpha\,(1 + m_0)$$

$$L_1 = \int_{\beta}^{\varphi} \frac{w}{w'^2}\left\{1 + \frac{3 + 2m_0}{1 + m_0}\,[z(\varphi')\cos\varphi' - z'(\varphi')\sin\varphi']\right\} d\varphi'$$

$$L_2 = \int_{\beta}^{\varphi} \frac{w}{w'^2}\, d\varphi', \qquad L_3 = \int_{\beta}^{\varphi} \frac{w}{w'^2}\, d\varphi' \int_{\varphi'}^{\varphi} \left(\frac{w^3}{w'^2}\right)_{\varphi'=\xi} d\xi$$

$$L_4 = \int_{\beta}^{\varphi} \frac{w}{w'^2}\, d\varphi' \int_{\varphi'}^{\varphi} \left(\frac{w^3}{w'^2}\right)_{\varphi'=\xi} d\xi \int_{\beta}^{\xi} \left(\frac{u}{w'^2}\right)_{\varphi'=\eta} d\eta \qquad (1.20)$$

$$L_5 = \int_{\beta}^{\varphi} \frac{w}{w'^2}\, d\varphi' \int_{\varphi'}^{\varphi} \left(\frac{w^3}{w'^2}\right)_{\varphi'=\xi} d\xi \int_{\beta}^{\xi} \left(\frac{w}{w'^2}\right)_{\varphi'=\eta} d\eta$$

The coefficients L_i are complicated functionals of the shape of the shock wave z. In some cases, however, for example when the wave is plane, the quadratures are readily computed and all L_i are represented by finite expressions. This situation is utilized later in the construction of the solution. Let us examine now the structure of Eq. (1.20) in more detail. The right side of this equation contains terms of higher order than the left side, and it may appear that they can be neglected. However, the totality of all terms in the left part of the equation is identically equal to zero at the point of conjunction. Therefore, near this point the left side may have the same order of smallness as the right side which contains the higher derivative. The presence of the latter allows to make a smooth conjunction of the plane wave with the curved wave(*). In this case,

*) This situation was not noted in any of the previous investigations, including the last papers [3, 4]. Therefore, attempts to construct solutions on the basis of an equation which differs from (1.20) by having a zero in the right side, ended without success.

according to (1.20), a discontinuity of curvature arises at the point A (Fig. 2). It should be noted that although the system (1.5) and boundary conditions (1.6) and (1.7) allow an error in terms of the order higher than the first from the very beginning, the most accurate possible solution of the approximate equations is sought. It is evident from the analysis of (1.20) that in the flow field there are regions in which the principal terms of their combinations become zero, and the behavior of the solution is determined by small additional terms. It is not always possible to determine in advance where and which of the small terms will turn out to be substantial, especially when the flow is complicated, as for example in the vicinity of the point of conjunction. In this case the retaining of all terms permits the detection of fine effects which make a major contribution in the region where the flows are matched.

Let us turn directly to the finding of solution (1.20). We shall seek the equation for the surface of the shock wave in the region $0 \leqslant \varphi \leqslant \varphi_2$ in the form of a series

$$z(\varphi) = \sum_{n=0}^{\infty} \frac{z^{(n)}(\varphi_2)}{n!} (\varphi - \varphi_2)^n \tag{1.21}$$

The values of functions $z(\varphi_2)$ and $z'(\varphi_2)$ are given by the condition of smooth conjunction with the homogeneous flow. At the point φ_2 the coefficients L_i are continuous and known. Therefore it is not difficult to establish a connection between $z'''(\varphi_2)$ and $z^{(4)}(\varphi_2)$ from (1.20). By successive differentiation of (1.20) we can find analogous relationships between $z^{(4)}(\varphi_2)$, $z^{(5)}(\varphi_2)$, etc. As a result the series (1.21) depends on only one arbitrary constant which can be easily found from the condition $z'(0) = 0$ in the plane of symmetry.

We can show that in the constructed solution the regions of homogeneous potential and vortex flows are matched in a continuous manner.

Although the curvature of the shock wave suffers a discontinuity at the point of conjunction, the gasdynamic characteristics still remain continuous. For simplicity we limit ourselves to the case of a flat wing ($z^\circ \equiv 0$) and examine the behavior of pressure in the vicinity of the line $\varphi = \varphi_2$. It is evident from Eq. (1.4) that the discontinuity can arise only from the second term of p_2. To the right of point p_2 expression (1.19) gives $p_2 = 0$; to investigate the pressure from the left we apply the relationship (1.26), then we again obtain $p_2 = 0$. Consequently the pressure is continuous inside the flow region.

The second characteristic which depends on the curvature of the wave is the velocity component v. Omitting the calculations we can make the conclusion that within the required accuracy to terms of ε^2, the velocity v remains continuous.

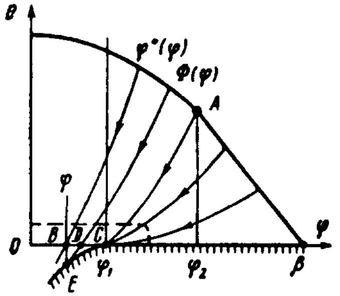

Fig. 3

Another important point is the obtaining of the topological picture of stream surfaces from the solution. Let us study the distribution of streamlines on a sphere $r = \text{const}$ after they have passed through a plane shock. According to (1.10) all these lines have a critical point $(0, \varphi_1)$ on the node type (Fig. 3). This node is located at a very short distance from the plane of symmetry $(\sim \varepsilon \mathrm{tg}^2\alpha)$, however, it does not coincide with it. Weak curvature of stream surfaces to the left of the characteristic $\varphi = \varphi_2$ is due to the change in function R.

Examination of streamlines which pass through the curvilinear shock reveals the possibility that these lines may intersect with the surface of the wing OC which in this region becomes singular (Fig. 3). It turns out that for $\varphi < \varphi_1$ (point B in Fig. 3) a range of values $\Phi \leqslant \varphi' \leqslant \beta$ exists for which the quantity θ assumes negative value. Thus, the solution encompasses a region on the other side of the wing down to the curved streamline EC. Formally, this region may be retained, if a real significance is attributed to all expressions only for $\theta > 0$. However, if a new function $\varphi^\circ(\varphi)$ $(0 \leqslant \varphi \leqslant \varphi_1)$ which is determined by the equation

$$\int\limits_{\beta}^{\varphi^\circ(\varphi)} \frac{w}{w'^2} R \, d\varphi' = 0$$

is introduced and the variation of the variable φ' is limited to the interval $\varphi \leqslant \varphi' \leqslant \varphi^\circ(\varphi)$, then the region $\theta < 0$ is excluded automatically. In this case the limit of integration β is replaced by the quantity $\varphi^\circ(\varphi)$, and the form of some expressions changes on differentiation. It is understandable that both approaches give one and the same result and lead to the necessity of satisfying the condition of flow on the line OC. Through the estimation of normal velocity it will be demonstrated that this condition is satisfied. If we take into account that $v = w\theta_\varphi$, $\theta_\varphi \sim \varepsilon/\varphi^\circ$ and $w \sim \varepsilon\varphi^\circ$, we obtain that $v \sim \varepsilon^2$, and the condition of flow is satisfied within the required accuracy.

In this connection it is necessary to note that although the constructed solution satisfies all equations and boundary conditions in the region adjacent to the section OC, the accuracy of the approximating system declines. Here, just as in the theory of hypersonic conical flows, it is necessary to take into account the entropy layer. As a result, in the region bounded in Fig. 3 by a dashed line, there will be a different velocity distribution and the singularities on the section OC will disappear.

The solution of Eq. (1.20) in the form of a series (1.21) is convenient because of its simplicity for carrying out concrete computations. However, it does not present the possibility to establish the form of the dependence of the solution on the parameter ε. In order to analyze the desired dependence, we can take advantage of the perturbation method. For $\varepsilon = 0$ the solution $z \equiv 0$ satisfies the boundary conditions and Eq. (1.20). If it is taken as the zeroth approximation of the solution for $\varepsilon \neq 0$, all integrals L_i can be computed. As a result Eq. (1.20) becomes a differential equation and for small φ reduces to Euler's equation of the third order. The solution of the latter contains terms of the form $\exp(\varepsilon^{-1/2} \ln \varphi)$ which indicate the nonlinear character of the dependence on ε. The direct substitution of this solution into (1.19) permits to establish that terms δ_1 and δ_2 correspond to estimates presented above.

2. The general relationships obtained above can be applied to the flow over a triangular plate at an angle of attack.

The angle at the apex of the plate is designated by 2β. For simplicity the Mach number will be assumed to be equal to infinity ($m_0 = 0$). In this case the function $z^\circ \equiv 0$ and the coefficients L_i, computed along the plane wave $z = a \sin(\beta - \varphi)$, are determined by the following relationships:

$$L_1 = \frac{1 + 2a \sin\beta}{\Delta_1} \frac{\sin(\beta - \varphi)}{\sin(\beta - \varphi_1)}, \qquad L_2 = \frac{\sin(\beta - \varphi)}{\Delta_1 \sin(\beta - \varphi_1)}$$

$$L_4 = \frac{\sin 2(\varphi - \varphi_1)}{2\Delta_1} d, \qquad L_5 = \frac{-\sin^2(\varphi - \varphi_1)}{\Delta_1} d \tag{2.1}$$

$$\Delta_1 = \cos\alpha\,(1 - 2a\,\mathrm{tg}^2\alpha\sin\beta)^{1/2},\quad \varphi_1 = a\,\mathrm{tg}^2\alpha\,\mathrm{ctg}\,\beta$$

$$d = \frac{5}{6}\,\mathrm{ctg}^3(\beta - \varphi_1)\sin^3(\varphi - \varphi_1) + \frac{1}{2}\,\frac{\mathrm{ctg}\,(\beta - \varphi_1)\sin(\beta - \varphi)\sin 2(\varphi - \varphi_1)}{\sin(\beta - \varphi_1)} -$$
$$- \frac{1}{3}\cos^3(\varphi - \varphi_1)$$

In this problem the coefficient L_3 drops out (the wing is plane); the coordinate φ_2 is determined by the line of intersection of the Mach cone for the homogeneous flow behind the shock wave, with the wing surface. Its value is found from the expression

$$\sin(\varphi_2 - \varphi_1) = \frac{1}{\Delta_1}\left(\varepsilon\,\frac{1+\varepsilon}{1-\varepsilon}\,p^*\right)^{1/2} \tag{2.2}$$

Equation (2.2) shows in particular that $\varphi_2 \sim \sqrt{\varepsilon}$. Let us substitute now the equation of the plane wave into (1.20); we obtain

$$a = \varepsilon\,\frac{1 + \varepsilon\sec^2\alpha}{\sin(\beta - \varphi_1)} \tag{2.3}$$

and in physical coordinates the equation of the plane shock assumes the form

$$\theta^* = \varepsilon\,\mathrm{tg}\,\alpha\,\frac{(1 + \varepsilon\sec^2\alpha)\sin(\beta - \varphi)}{\sin(\beta - \varphi_1)}$$

The pressure in the homogeneous flow behind the shock is determined by the formula

$$p = p^* = \sin^2\alpha + \varepsilon\left[\frac{\sin 2\alpha\sin\beta\,\mathrm{tg}\,\alpha}{\sin(\beta - \varphi_1)}(1 + \varepsilon\sec^2\alpha) - \sin^2\alpha\right] \tag{2.4}$$

The curvilinear region of the wave is found with the aid of series (1.21). As a preliminary step, the coefficients L_i calculated from Eqs. (2.1) are substituted into Eqs.(1.20) for $\varphi = \varphi_2$. Then at the point φ_1 the following relationship exists between the derivatives:

$$z''' = -(z_2'' + z_2)\,k - z_2', \qquad k = 2\,\mathrm{ctg}\,(\varphi_2 - \varphi_1) + \mathrm{ctg}\,\varphi_2$$
$$z_2 = z(\varphi_2) \tag{2.5}$$

Analogous relationships can be established also between derivatives of higher order $z^{(4)}$, $z^{(5)}$ etc. if successive differentiation of (1.20) is carried out.

For simplicity let us limit the series (1.21) to four terms. In this approximation, utilizing the boundary condition $z'(0) = 0$, we obtain the following expression for the second derivative:

$$z_2'' = \frac{z_2'(2 - \varphi_2^2) - k\varphi_2^2 z_2}{\varphi_2(2 + k\varphi_2)} \tag{2.6}$$

Equation (2.6) allows to determine directly the order z_2''. In fact, $\varphi_2 \sim \sqrt{\varepsilon}$, z_2 and $z_2' \sim \varepsilon$, therefore, z_2'' is $\sim \sqrt{\varepsilon}$. From (2.5) we find correspondingly that $z_2''' \sim 1$. It is not difficult to establish that the shock wave is convex upward (the sign of $z_2'' < 0$). The relationship (2.5) permits to represent the series (1.21) in the following form:

$$z(\varphi) = z_2 + z_2'(\varphi - \varphi_2) + {}^1/_2\,z_2''(\varphi - \varphi_2)^2 - {}^1/_6\,[(z_2 + z_2'')\,k +$$
$$+ z_2'](\varphi - \varphi_2)^3 + \ldots \tag{2.7}$$

The second derivative is eliminated by means of (2.6). Having established the shape of the shock wave, the remaining characteristics of the flow field are readily found.

Let us examine some data on pressure distribution on the surface of the wing. In Fig. 4 the pressure coefficient is shown as computed from the first approximation taking into account p_2 (dashed line). A comparison is made with corresponding quantities obtained in [3] by method of finite differences and Newton's theory (dashed line). In the same figure results are presented for comparison with experimental data [6] at $M = 5$. It is evident from the location of curves that agreement of data is sufficiently good.

3. As the second example we shall examine the flow over a triangular wing with a V-shaped cross section.

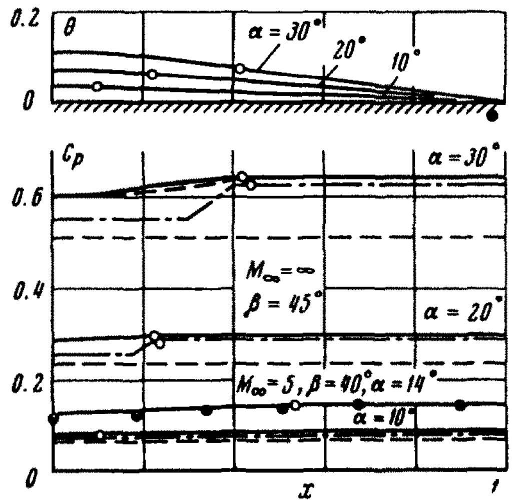

Fig. 4

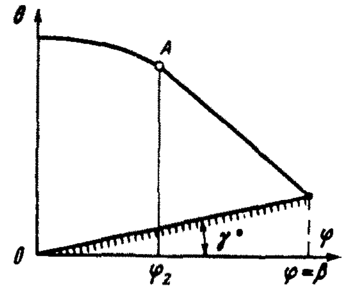

Fig. 5

The half-angle at the apex of the wing is designated by β. The angle measured in the plane of the cross section is designated by γ (Fig. 5). The equation of the transverse contour z° is determined by the expression $z^\circ = \operatorname{ctg}^2 \alpha\gamma \sin\varphi$. It is assumed that $\gamma \ll \varepsilon$. Coefficients L_i computed for the plane region of the shock wave remain equal to expressions (2.1). In this case the equation of the wave is

$$z = z^\circ(\varphi) + a \sin(\beta - \varphi)$$

The coefficient L_3 again drops out. Equations (2.2) and (2.3) remain unchanged with the required accuracy. As a result the equations of the plane shock assume the following form in physical coordinates

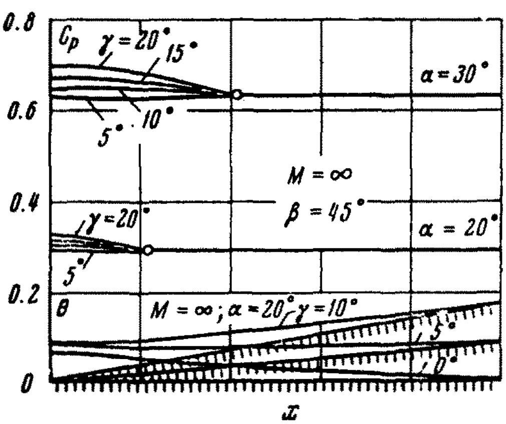

Fig. 6

$$\theta^* = \gamma \sin\varphi + \frac{\varepsilon \operatorname{tg}\alpha\,(1 + \varepsilon \sec^2\alpha)\sin(\beta - \varphi)}{\sin(\beta - \varphi_1)} \qquad (3.1)$$

Let us determine now the curvilinear region of the wave. At the point A, Eq. (1.20) and the coefficients L_i coincide with the case of the plane wing. Therefore, relations (2.5) and (2.6) can also be used for the V-shaped wing. Taking into account the previous statement, the series (2.7) will represent the shock wave if the values z_2 and z_2' are determined from (2.8). The qualitative analysis of the solution connected with the discontinuity in curvature, the entropy layer, etc., can be omitted because it does not contain new aspects as compared with the analysis which was carried out for the plane wing.

Computations of pressure distribution on the wing, and the shape of the shock wave are presented in Fig. 6 for several values of the angle γ and the angle of attack. The graphs, which are constructed form the first approximation, permit to draw the conclusion that the pressure change is insignificant along the span of the wing. The principal change is observed near the plane of symmetry, where for decreasing γ an increase in the values of pressure takes place.

We note that the theory is applicable to flows without internal shocks, therefore the angle γ can change in a relatively narrow range.

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ON THE ENTROPY LAYER IN HYPERSONIC FLOWS WITH SHOCK WAVES WHOSE SHAPE IS DESCRIBED BY A POWER FUNCTION

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Many results in the theory of hypersonic flows past slender bodies are based on the analogy with unsteady flows in a space with one fewer dimensions. This analogy was developed by the authors of papers [1-4]. However, its use for calculating gas parameters near the surface of a body often entails considerable errors. For the purpose of accurate determination of flow characteristics throughout the domain, the authors of [5-9] developed the notion of a high-entropy layer which contains estimates of the required quantities along the streamlines intersecting the front of the strong bow shock wave. Entropy layer methods have proved especially convenient in dealing with inverse aerodynamic problems in which the position of the shock wave is assumed to be given and the shape of the body must be determined in the course of solution. Specifically, the authors of [6-9] investigated the problem of the body shape associated with the motion of a gas due to an intense explosion.

The analysis of the results of Sychev and Yakura carried out by the authors of [10]